CLUSTER CATEGORIES, m-CLUSTER CATEGORIES AND DIAGONALS IN POLYGONS

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Abstract. The goals of this expository article are on one hand to describe how to construct (m-) cluster categories from triangulations (resp. from $m+2$-angulations) of polygons. On the other hand, we explain how to use translation quivers and their powers to obtain the m-cluster categories directly from the diagonals of a polygon.

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Introduction

This expository article is the expanded version of a talk given at the conference at the Grenoble summer school "Geometric methods in representation theory" in July 2008. The goal of this talk was to explain how cluster categories and m-cluster categories can be described via diagonals and so-called m-diagonals in a polygon. And then how the latter can actually be described using the power of a translation quiver. The first section gives a very brief introduction to the theory of cluster algebras and cluster categories. It also introduces the notations used in the article. In Section 2, we explain the notions of a quiver given by the diagonals in a polygon and of the one given by m-diagonals. The results in this section are mainly due to Caldero-Chapoton-Schiffler ([CCS06]), to Schiffler ([S06]) and to Baur-Marsh ([BaM08], [BaM07]). In the last section, we introduce the concept of the power of a translation quiver. Here, the results are from [BaM08], [BaM07] and from the masters thesis of C. Ducrest ([D08]).

1. Cluster algebras and cluster categories

Cluster algebras were introduced by Fomin and Zelevinsky ([FZ1]) in order to provide an algebraic framework for the phenomena of total positivity and for the canonical bases of the quantized universal enveloping algebras.

2000 *Mathematics Subject Classification.* Primary: Secondary:
Key words and phrases. Cluster category, m-cluster category, diagonals, translation quiver.

We briefly illustrate the notion of total positivity: An $n \times n$ matrix is called *totally positive* if all its minors are positive. Originally, the term was used to describe matrices with non-negative minors: these matrices are nowadays called *totally non-negative*. In the 1930s, Gantmacher-Krein and I. Schoenberg have independently started investigating such matrices. One of the motivations was to estimate the number of real zeroes of a polynomial.

Gantmacher showed that totally non-negative matrices have different real eigenvalues. The interest in total positivity was renewed in the 1990s when G. Lusztig extended the notion to reductive algebraic groups, cf. [Lu94].

Example 1.1. To illustrate the notion on a (non-reductive) example, let us consider the group of 3 by 3-matrices with 1's on the diagonal and zeroes below. If $U = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is such a matrix then U is totally positive if $a, b, c > 0$ and $ac - b > 0$. One can check that it is actually enough to require $a > 0,\ ac - b > 0$ and $b > 0$: the condition $c > 0$ will follow automatically. Equivalently, the condition $a > 0$ can be dropped and is automatically satisfied by the remaining conditions. So there is only a certain number of minors that need to be checked.

Furthermore, if from the set $\{a > 0, ac - b > 0, b > 0\}$ of conditions the first is omitted then we may replace it exactly with one other condition, namely with $c > 0$, to obtain total positivity of the matrix.

More generally, one shows that the minimal sets of minors to check all have the same cardinality. And it is often the case that if you remove one minor from such a minimal set, there exists exactly one other minor to replace it with.

1.1. Cluster algebras.

A cluster algebra $\mathcal{A} \subset \mathbb{Q}(u_1, \dots, u_n)$ of rank n is an algebra with possibly infinitely many generators. These generators are called cluster variables; they are arranged in overlaping sets of the same cardinality n, the *clusters*. There are relations between the cluster variables, encoded in an $n \times n$ matrix, the *mutation matrix*. Through mutation, one element of a cluster is exchanged by exactly one other element and this exchange process is prescribed by the exchange matrix.

If there are only finitely many generators, the cluster algebra is of finite type. Finite type cluster algebras have been classified by Fomin and Zelevinsky ([FZ03]). Their classification describes the finite type cluster algebras in terms of Dynkin diagrams.

More concretely: a seed is a pair $(\underline{x}, M)$ where $\underline{x} = \{x_1, \dots, x_n\}$ is a basis of $\mathbb{Q}(u_1, \dots, u_n)$ and $M = (M_{ij})_{ij}$ is a sign skew symmetric $n \times n$-matrix with integer entries, called the exchange matrix. That means that the sign of M_{ij} is the opposite of the sign of M_{ji}.

Then one defines an involutive map μ_k (for $k \in \{1, \dots, n\}$) on the seeds, called the mutation in direction of k, through $\mu_k(\underline{x}) = (x_1, \dots, \widehat{x}_k, x'_k, \dots, x_n)$ where x'_k is given by the relation

$$x_k \cdot x'_k = \prod_{\substack{x_i \in \underline{x} \\ M_{ik} > 0}} x_i^{M_{ik}} + \prod_{\substack{x_i \in \underline{x} \\ M_{ik} < 0}} x_i^{-M_{ik}}$$

In a similar way, one defines M' by

$$M'_{ij} := \begin{cases} -M_{ij} & \text{if } i = k \text{ or } j = k \\ M_{ij} + \frac{1}{2}(|M_{ik}| M_{kj} + M_{ik} |M_{kj}|) & \text{otherwise.} \end{cases}$$

and thus obtains $(\underline{x'}, M')$ as $\mu_k((\underline{x}, M))$ (the matrix M' is also a sign skew symmetric $n \times n$-matrix over $\mathbb{Z}$). For more details we refer to Section 1 of the survey

article [BuM06] of Buan-Marsh. The x_i obtained through successive mutations are the so-called cluster variables. The cluster algebra $\mathcal{A} = \mathcal{A}(\underline{x}, M)$ is then defined as the algebra generated by the cluster variables. There can be infinitely many of them. Fomin-Zelevinsky have shown in [FZ4] that $\mathcal{A}$ lies in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ (Laurent-phenomenon). First examples of cluster algebras are coordinate rings of SL_2, SL_3.

The field of cluster algebras is a young and very dynamic field. Since its first introduction, there have been many different directions in its development. We only mention a few connections to other areas (in parentheses: the objects corresponding to the cluster variables): the theory of Teichmüller spaces (Penner coordinates), see work of Fock-Goncharov, [FG06] and [FG07]; the representation theory of finite dimensional algebras (tilting modules), cf. [BMRRT05]), triangulations of surfaces (diagonals), see the work [FST08] of Fomin-Shapiro-Thurston.

1.2. **Cluster categories.** Cluster categories were introduced independently in the work [BMRRT05] of Buan-Marsh-Reineke-Reiten-Todorov, and by Caldero-Chapoton-Schiffler, [CCS06] to provide a categorification of the theory of cluster algebras.

We will use the approach of [BMRRT05] to describe cluster categories and will consider the approach of [CCS06]later, cf. Section 2.

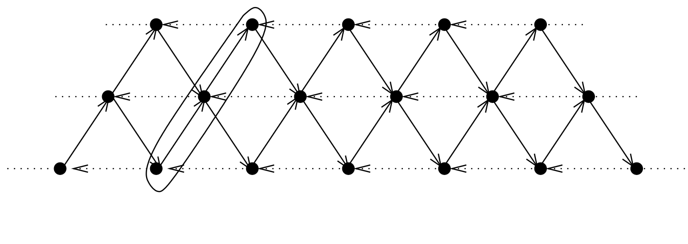
Let Q be a simply-laced Dynkin quiver (i.e. a quiver whose underlying graph is of type A, D or E). Let k be an algebraically closed field and kQ the path algebra of Q (for more details, we refer to the lecture notes of M. Brion, [B08] in the same volume). Take the bounded derived category $\mathcal{D}^b(kQ)$ of finitely generated kQ-modules (for details on $\mathcal{D}^b(kQ)$ we refer to [H88]) with shift denoted by [1] and Auslander-Reiten translate denoted by τ. By Happel ([H88]), the category $\mathcal{D}^b(kQ)$ is triangulated, Krull-Schmidt, and has almost split sequences. To understand the category $\mathcal{D}^b(kQ)$ it is helpful to study its Auslander-Reiten quiver: The Auslander-Reiten quiver of a category is a combinatorial tool which helps understanding the category. Its vertices are by definition the indecomposable modules up to isomorphism and the number of arrows between two points are given by the dimension of the space of irreducible maps between two representatives of the corresponding modules.

We now associate a quiver $\mathbb{Z}Q$ to Q. Its vertices are (n, i) for $n \in \mathbb{Z}$, and where i a vertex of Q. For every arrow $i \to j$ in Q there are arrows $(n, i) \to (n, j)$ and $(n, j) \to (n+1, i)$ in $\mathbb{Z}Q$. So $\mathbb{Z}Q$ has the shape of a $\mathbb{Z}$-strip of copies of Q. Together with the map $\tau : (n, i) \to (n-1, i)$ ($n \in \mathbb{Z}$, i a vertex of Q), $\mathbb{Z}Q$ is a stable translation quiver as defined by Riedtmann (see [Rie90]). For a precise definition of stable translation quivers we refer the reader to Section 3 below. We illustrate $\mathbb{Z}Q$ in Example 1.2. Happel has shown in [H88], that the Auslander-Reiten quiver $\mathrm{AR}(\mathcal{D}^b(kQ))$ of $\mathcal{D}^b(kQ)$ is just $\mathbb{Z}Q$. In particular, the category $\mathcal{D}^b(kQ)$ is independent of the orientation of Q.

Example 1.2. Let Q be a quiver of type A_3,

$$Q : \quad \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$$

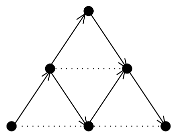
Then, $\mathbb{Z}Q$ has the shape



with one copy of the quiver Q highlighted to show how it appears inside $\mathbb{Z}Q$. The dotted arrows indicate the Auslander-Reiten translate τ which sends each vertex to its leftmost neighbor. It is an auto-equivalence of the Auslander-Reiten quiver.

On the other hand, the Auslander-Reiten quiver of the module category kQ-mod of finitely generated kQ-modules looks like a triangle:

AR(kQ-mod):



Observe that the infinite quiver $\mathbb{Z}Q$ can be viewed as being covered by copies of the Auslander-Reiten quiver of the module category kQ-mod, with additional arrows and dotted arrows introduced to connect the copies of the triangle of kQ-mod. With this picture in mind, we can describe the shift [1] on $\mathrm{AR}(\mathcal{D}^b(kQ))$: it sends each vertex to the "same" vertex in the next copy of the triangle AR(kQ-mod) to the right.

Back to the general situation, where Q is of simply-laced Dynkin type. The shift [1] then is the auto-equivalence of $\mathrm{AR}(\mathcal{D}^b(kQ))$ which sends a vertex to the corresponding vertex in the next copy of the Auslander-Reiten quiver of the module category kQ-mod and the translation τ sends a vertex to its leftmost neighbor. As an abbreviation, we write F for the auto-equivalence $\tau^{-1} \circ [1]$ of $\mathcal{D}^b(kQ)$. Now we are ready to define the cluster category associated to Q.

Definition. The *cluster category* $\mathcal{C} := \mathcal{C}_Q := \mathcal{D}^b(kQ)/F$ *of type* Q is the orbit category whose objects are the F-orbits of objects of $\mathcal{D}^b(kQ)$ and whose morphisms are given as follows:

$$\mathrm{Hom}_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) = \bigsqcup_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(kQ)}(F^i X, Y)$$

where $\widetilde{X}$ and $\widetilde{Y}$ are representatives of the F-orbits through X and Y respectively.

Note that for any pair of objects X, Y of $\mathcal{D}^b(kQ)$ there are only finitely many i such that $\mathrm{Hom}_{\mathcal{D}^b(kQ)}(F^i X, Y)$ is non-zero. The cluster category is Krull-Schmidt ([BMRRT05]), triangulated and Calabi-Yau of dimension 2 ([Ke05]).

The connection between $\mathcal{C}_Q$ and the cluster algebra of the same type is given by the following result.

Theorem 1.3 ([BMRRT05]). *There is a bijection between the cluster variables of the cluster algebra of type A_n (resp. D_n, E_n) and indecomposable objects of $\mathcal{C}_Q$ where Q is of type A_n (resp. D_n, E_n).*

To understand the cluster categories better, we consider its Auslander-Reiten quiver. By definition, it has the form of one copy of the module category, together with a copy of the quiver Q (with additional arrows, dotted arrows), as illustrated in types A and D below (Figures 1 and 2). In particular, it is a finite quiver. In the pictures of Figures 1 and 2 we have repeated one copy of the quiver Q to indicate how the quivers are glued together: both quivers wrap around. The Auslander-Reiten quiver of type A_n can be viewed as lying on a Möbius strip and the one of type D_n as lying on a cylinder.

Let G be the underlying graph of Q, G of Dynkin type A, D or E. We recall that a famous result of P. Gabriel establishes a bijection between the indecomposable objects of kQ-mod (up to isomorphism) and the positive roots of the Lie algebra of type G. For a recent description of this result we refer to Section 5 of the lecture notes [Kr07] of H. Krause.

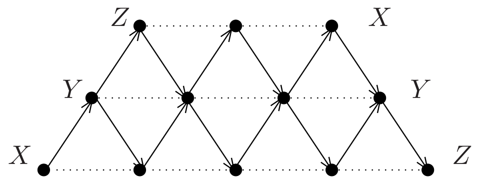


Figure 1. The Auslander-Reiten quiver of $\mathcal{C}$ for A_3

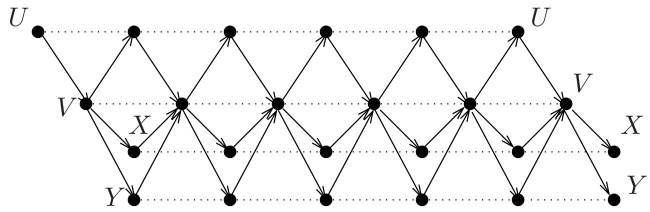


Figure 2. The Auslander-Reiten quiver of $\mathcal{C}$ for D_4

There exists an analogous result for cluster categories. To state it we need to enlarge the set of roots considered: we add the negatives of the simple roots. An *almost positive root* of a Lie algebra of type G is a positive root or the negative of a simple root.

Buan et al. have shown in [BMRRT05] that there is a bijection between the indecomposable objects of the cluster category $\mathcal{C}_Q$ and the almost positive roots of the Lie algebra of type Q.

1.3. **The m-cluster category.** In 2005, Keller ([Ke05]) has introduced the m-cluster categories as a natural generalisation of the cluster categories. Again, let Q be a quiver whose underlying graph is of Dynkin type A, D or E. Let $[1]$ be the shift and τ the Auslander-Reiten translate as before. Let F_m be the auto-equivalence $\tau^{-1} \circ [m]$ of $\mathcal{D}^b(kQ)$, for $m \geq 1$.

Definition. The *m-cluster category* $\mathcal{C}^m := \mathcal{C}_Q^m := \mathcal{D}^b(kQ)/F_m$ *(of type Q)* is the orbit category with objects the F_m-orbits of objects of $\mathcal{D}^b(kQ)$ and with morphisms $\mathrm{Hom}_{\mathcal{C}^m}(\widetilde{X}, \widetilde{Y}) = \coprod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(kQ)}(F_m^i X, Y)$ where $\widetilde{X}$ and $\widetilde{Y}$ are representatives of the F_m-orbits of X and Y respectively.

Note that the Auslander-Reiten quiver of $\mathcal{C}^m$ thus consists of m copies of the Auslander-Reiten quiver of the module category kQ-mod and additionally, of a copy of the quiver Q, connected with additional (dotted) arrows. In the case of A_n, we observe that if m is odd, the Auslander-Reiten quiver of $\mathcal{C}_Q^m$ lies on a Möbius strip, whereas if it m is even, it lies on a cylinder. As an example, Figure 3 shows the Auslander-Reiten quiver of the 2-cluster category of type A_2. Again, we have repeated a slice of the quiver to indicate how it wraps around.

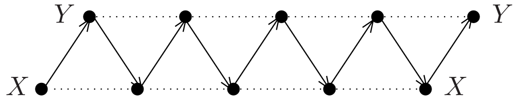


Figure 3. The Auslander-Reiten quiver of $\mathcal{C}^2$ for A_2

Remark. The m-cluster categories have very nice properties, analogously to the properties of the cluster categories: $\mathcal{C}^m$ is Krull-Schmidt ([BMRRT05]), triangulated and Calabi-Yau of dimension $m+1$ ([Ke05]).

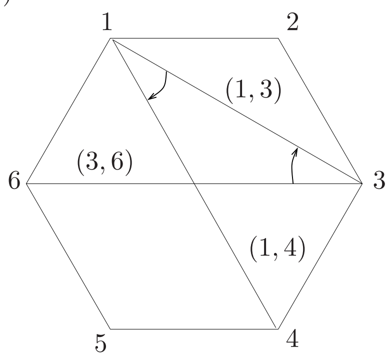
2. Polygons and diagonals

In the first part of this section we present the approach of Caldero, Chapoton and Schiffler, [CCS06], who described the cluster category of type A_n in terms of the diagonal of a polygon. This was later adapted to type D_n by Schiffler in [S06], using a punctured n-gon. In the second part, we explain how to describe the m-cluster category of type A_n in terms of so-called m-diagonals in a polygon.

2.1. **Quiver of diagonals.** Let Π be a polygon with $n+2$ vertices, labeled clockwise by $\{1, 2, \ldots, n+2\}$. To it, we associate a quiver $\Gamma(n, 1)$ as follows:

The vertices of $\Gamma(n, 1)$ are the diagonals $\{(i, j) \mid 1 \le i, j \le n+2,\ |i-j| > 1\}$ of Π. The arrows are $(i, j) \to (i, j+1)$ and $(i, j) \to (i+1, j)$, where $i+1$ and $j+1$ are taken modulo $n+2$, provided the image is also a diagonal of Π. Furthermore, we define a bijection τ on the vertices of $\Gamma(n, 1)$ as follows: $\tau : (i, j) \mapsto (i-1, j-1)$ (again, taking $i-1$ and $j-1$ modulo $n+2$). The quiver $\Gamma(n, 1)$ together with this map τ is a stable translation quiver (cf. definition in Section 3 below).

Example 2.1 (Hexagon)**.** Let us illustrate this in the case $n = 4$.



The translation quiver $\Gamma(4, 1)$ obtained from the hexagon is:

$$
\begin{tikzcd}[column sep=small]
 & & (1,5) \arrow[dr] & & (2,6) \arrow[ll, dotted] \arrow[dr] & & (1,3) \arrow[ll, dotted] \arrow[dr] & & \\
 & (1,4) \arrow[ur] \arrow[dr] & & (2,5) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (3,6) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (1,4) \arrow[ll, dotted] \arrow[dr] & \\
(1,3) \arrow[ur] & & (2,4) \arrow[ll, dotted] \arrow[ur] & & (3,5) \arrow[ll, dotted] \arrow[ur] & & (4,6) \arrow[ll, dotted] \arrow[ur] & & (1,5) \arrow[ll, dotted]
\end{tikzcd}
$$

We have repeated the first slice $(1, 3) \to (1, 4) \to (1, 5)$ at the end to indicate how the quiver wraps around.

Observe that the quiver $\Gamma(4, 1)$ is equal to the Auslander-Reiten quiver of the cluster category of type A_3 (Figure 1). More generally, one can show that the quiver of diagonals of an $n+2$-gon encodes the cluster category of type A_{n-1}:

Theorem 2.2 ([CCS06], Section 2)**.** *Let Q be a quiver of Dynkin type A_{n-1}. Then the Auslander-Reiten quiver of $\mathcal{C}_Q$ is isomorphic to the quiver $\Gamma(n, 1)$ of diagonals in an $n+2$-gon.*

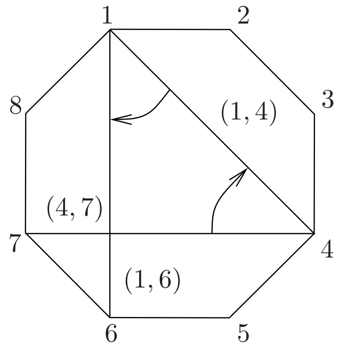
As a consequence of this, the cluster category $\mathcal{C}_Q$ is equivalent to the additive category generated by the mesh category of the stable translation quiver $\Gamma(n, 1)$ of diagonals of an $n+2$-gon. For details, we refer the reader to [CCS06] and [BaM08].

Remark. The cluster category of type D_n can be modelled in a similar way if we use a punctured n-gon. This has been done by Schiffler in [S06] using arcs in the punctured polygon.

Remark. Every maximal collection of non-crossing diagonals of a polygon Π (punctured or not) is a triangulation of Π. All maximal collections have the same number of elements, this number is an invariant of Π. It is called the *rank of the polygon*. The rank of an $n+2$-gon is $n-1$, the rank of a punctured n-gon is n. This leads us back to cluster algebras - for the connection between (punctured) polygons of rank n and cluster algebras of type A_n (of type D_n respectively) we refer the reader to [FST08].

2.2. **Quiver of m-diagonals.** In a similar way, the m-cluster categories can be modelled using a certain quiver $\Gamma(n,m)$. We will now explain how this works. Let Π be an $nm+2$-gon. The vertices of $\Gamma(n,m)$ are the m-diagonals, i.e. the diagonals of the form $(1,m+2)$, $(1,2m+2)$, etc., where vertices are taken modulo $nm+2$. More precisely, an m-diagonal divides Π into an $mj+2$-gon and an $m(n-j)+2$-gon (for $1 \leq j \leq \frac{n-1}{2}$). The arrows send (i,j) to $(i,j+m)$ and to $(i+m,j)$ whenever the image is also an m-diagonal. Furthermore, we define a translation τ_m on $\Gamma(n,m)$ which sends (i,j) to $(i-m,j-m)$ (taking vertices modulo $nm+2$). Then $\Gamma(n,m)$ is also a stable translation quiver. With $m=1$ we just recover the case of the usual diagonals as described above.

Example 2.3. Let $m=2$ and $n=3$; Π is an octagon. Its 2-diagonals are of the form $(1,4)$, $(1,6)$, $(2,5)$, etc.



Then the quiver $\Gamma(3,2)$ is:

$$\begin{tikzcd}
 & (1,6) \arrow[dr] & & (3,8) \arrow[ll, dotted] \arrow[dr] & & (2,5) \arrow[ll, dotted] \arrow[dr] & & (4,7) \arrow[ll, dotted] \arrow[dr] & & (1,6) \arrow[ll, dotted] \\
(1,4) \arrow[ur] & & (3,6) \arrow[ll, dotted] \arrow[ur] & & (5,8) \arrow[ll, dotted] \arrow[ur] & & (2,7) \arrow[ll, dotted] \arrow[ur] & & (1,4) \arrow[ll, dotted] \arrow[ur] &
\end{tikzcd}$$

Here, we have also repeated the first slice $(1,4) \to (3,6)$ to indicate how the quiver wraps around.

Observe that the quiver $\Gamma(3,2)$ is just the Auslander-Reiten quiver of the 2-cluster category of type A_2. So the 2-diagonals in the octagon model the cluster category $\mathcal{C}_Q^2$ for Q of Dynkin type A_2. This holds more generally by the following result.

Theorem 2.4 ([BaM08])**.** *Let Q be a quiver of Dynkin type A_{n-1}, let $m \geq 1$. Then the Auslander-Reiten quiver of $\mathcal{C}_Q^m$ is isomorphic to the quiver $\Gamma(n,m)$ of m-diagonals in an $nm+2$-gon.*

Note that we recover Theorem 2.2 in the case $m=1$.

Remark. (1) Theorem 2.4 tells us that in order to understand the m-cluster category of type A_n it is enough to study $\Gamma(n,m)$.

(2) To model the m-cluster categories of type D_n, one defines so-called m-arcs in a punctured $nm-m+1$-gon and obtains a quiver $\Gamma_{\odot}(n,m)$. One can show that

this is the Auslander-Reiten quiver of $\mathcal{C}_Q^m$ where Q is of type D_n, cf. Theorem 3.6 in [BaM07].

The maximal collections of non-crossing m-diagonals in an $nm+2$-gon (resp. in a punctured $nm-m+1$-gon) correspond to $m+2$-*angulations* of the polygon. The number of elements in such a maximal collection is again an invariant of the polygon. It is equal to $n-1$ (resp. to n).

3. Powers of translation quivers

We now provide another way of obtaining m-cluster categories. It uses the concept of the power of a translation quiver which was introduced in [BaM08]. In order to explain this, let us give the precise definition of a translation quiver.

Definition. A *translation quiver* is a pair (Γ,τ) where Γ is a quiver, possibly with infinitely many vertices and arrows; τ is an injective map from a subset of the vertices of Γ to the vertices of Γ, such that the following holds: the number of arrows going from a vertex x to y equals the number of arrows from τy to x for all vertices x,y of Γ. The map τ is called the *translation of* (Γ,τ).

If τ is defined on all vertices (and thus bijective) then (Γ,τ) is a *stable* translation quiver.

We remark that in all examples of stable translation quivers appearing in this article, the number of arrows between 2 vertices is always at most 1.

We recall that a composition $x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m$ of m arrows $x_i \to x_{i+1}$ (where the x_i are vertices of Γ) is a *path of length* m. Such a path is said to be *sectional* if $\tau x_{i+1} \neq x_{i-1}$ for $i=1,\dots,m-1$ (for which τx_{i+1} is defined), cf. [Rin84].

Definition. Let (Γ,τ) be a translation quiver. The *m-th power* Γ^m *of* Γ is the quiver whose vertices are the same as the vertices of Γ and whose arrows are the sectional paths of length m of Γ.

One can show that if (Γ,τ) is a stable translation quiver, then the pair (Γ^m,τ^m) is also a stable translation quiver ([BaM08], Section 6). Note however that (Γ^m,τ^m) is not connected in general, even if (Γ,τ) is so. The following example illustrates this.

Example 3.1. Let Γ be the quiver $\Gamma(6,1)$ of diagonals in an octagon, let $m=2$. The quiver of the octagon has five rows, with first slice $(1,3)\to(1,4)\to(1,5)\to(1,6)\to(1,7)$:

$$\begin{tikzcd}[column sep=tiny]
 & & & & (1,7) \arrow[dr] & & (2,8) \arrow[ll, dotted] \arrow[dr] & & (1,3) \arrow[ll, dotted] \arrow[dr] & & & & \\
 & & & (1,6) \arrow[ur] \arrow[dr] & & (2,7) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (3,8) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (1,4) \arrow[ll, dotted] \arrow[dr] & & & \\
 & & (1,5) \arrow[ur] \arrow[dr] & & (2,6) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (3,7) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (4,8) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (1,5) \arrow[ll, dotted] \arrow[dr] & & \\
 & (1,4) \arrow[ur] \arrow[dr] & & (2,5) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (3,6) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (4,7) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (5,8) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (1,6) \arrow[ll, dotted] \arrow[dr] & \\
(1,3) \arrow[ur] & & (2,4) \arrow[ll, dotted] \arrow[ur] & & (3,5) \arrow[ll, dotted] \arrow[ur] & & (4,6) \arrow[ll, dotted] \arrow[ur] & & (5,7) \arrow[ll, dotted] \arrow[ur] & & (6,8) \arrow[ll, dotted] \arrow[ur] & & (1,7) \arrow[ll, dotted]
\end{tikzcd}$$

As before, we repeat the first slice at the end to indicate how the quiver wraps around.

The second power of $\Gamma(6,1)$ has three components. One containing the vertex $(1,3)$, one containing the vertex $(1,4)$ and the third containing the vertex $(2,4)$:

$$\begin{tikzcd}[column sep=small]
 & (1,6) \arrow[dr] & & (3,8) \arrow[ll, dotted] \arrow[dr] & & (2,5) \arrow[ll, dotted] \arrow[dr] & & (4,7) \arrow[ll, dotted] \arrow[dr] & & (1,6) \arrow[ll, dotted] \\
(1,4) \arrow[ur] & & (3,6) \arrow[ll, dotted] \arrow[ur] & & (5,8) \arrow[ll, dotted] \arrow[ur] & & (2,7) \arrow[ll, dotted] \arrow[ur] & & (1,4) \arrow[ll, dotted] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & (1,7) \arrow[dr] & & (1,3) \arrow[ll, dotted] \arrow[dr] & & \\
 & (1,5) \arrow[ur] \arrow[dr] & & (3,7) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (1,5) \arrow[ll, dotted] \arrow[dr] & \\
(1,3) \arrow[ur] & & (3,5) \arrow[ll, dotted] \arrow[ur] & & (5,7) \arrow[ll, dotted] \arrow[ur] & & (1,7) \arrow[ll, dotted]
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
 & & (2,8) \arrow[dr] & & (2,4) \arrow[ll, dotted] \arrow[dr] & & \\
 & (2,6) \arrow[ur] \arrow[dr] & & (4,8) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (2,6) \arrow[ll, dotted] \arrow[dr] & \\
(2,4) \arrow[ur] & & (4,6) \arrow[ll, dotted] \arrow[ur] & & (6,8) \arrow[ll, dotted] \arrow[ur] & & (2,8) \arrow[ll, dotted]
\end{tikzcd}$$

We observe that the component through the vertex $(1,4)$ is the same as the quiver $\Gamma(3,2)$ from Example 2.3. This is a property that holds in general, cf. Theorem 3.2. The other two components are isomorphic to the Auslander-Reiten quiver of the orbit category $\mathcal{D}^b(kQ)/[1]$ where Q is of Dynkin type A_3.

Theorem 3.2. *The quiver $\Gamma(n,m)$ is a connected component of $\Gamma(nm,1)^m$.*

From Theorem 3.2 one obtains that the m-cluster category of type A_n is a full subcategory of the additive category generated by the mesh category of $\Gamma(nm,1)$ (for a definition of the mesh category we refer the reader to [BaM08, Section 3]). In other words: in order to understand the m-cluster category, there is an alternative approach to the one presented in Subsection 2.2. Namely, we can consider the m-th power of the quiver given by the usual diagonals. The Auslander-Reiten quiver of the m-cluster category of type A_{n-1} appears as a connected component of $\Gamma(nm,1)^m$.

Remark. We can actually prove an analogous result as Theorem 3.2 for type D_n, cf. [BaM07]: Let $\Gamma_\odot(n,1)$ denote the quiver obtained from the arcs of a punctured n-gon (note the difference: here, the polygon has n vertices instead of $n+2$) and by $\Gamma_\odot(n,m)$ the quiver of m-arcs in a punctured $nm-m+1$-gon. Then the Auslander-Reiten quiver of the m-cluster category of type D_n is $\Gamma_\odot(n,m)$ and it is a connected component of $\Gamma_\odot(nm-m+1,1)^m$. For details we refer to Section 5 of [BaM07][1].

A natural question to ask at this point is what the other connected components of the mth power of $\Gamma(nm,1)$ are. Remarkably, this question is much easier in type D. There, we have a complete answer:

The connected components of the (restricted) mth power of the Auslander-Reiten quiver of the cluster category of type D_{nm-m+1} are exactly the union of the Auslander-Reiten quiver of the m-cluster category of type D_n with $m-1$ copies of the Auslander-Reiten quiver of $\mathcal{D}^b(\mathrm{A}_{n-1})/\tau^{nm-m+1}$, cf. Theorem 5.2 of [BaM07].

The difficulty arising in type A has to do with the additional symmetry of the Dynkin graph of type A, i.e. with the involution sending the first to the last node, the second to the second-to-the-last, etc.

Let now Q be of Dynkin type A_n. For odd m, C. Ducrest([D08]) was recently able to answer the question of the components of the mth power. For even m, she provides a partial description. Her result is the following (see Section 4 of [D08]):

[1]to be precise: the arrows in the m-power arise from *restricted* sectional paths of length m, and we are taking the restricted mth power.

Theorem 3.3.

$$\Gamma(nm,1)^m = \Gamma(n,m) \quad \bigsqcup \quad \bigcup_{i=1}^{t} \Gamma_i$$

where Γ_i is the Auslander-Reiten quiver of an orbit category of $\mathcal{D}^b(A_n)$ under an auto-equivalence of the form $\tau^{-s} \circ [r]$ for some t and for some s, r where we can assume $s < n$.

Furthermore, if m is odd then $r = (m-1)/2$ and $s = \frac{1}{2}((m-1)(n-1)) + 1$.

The even case is more complicated. One can show that for even m, we have $m/2 \leq r \leq m$. Example 3.1 above shows that $r = m$ does occur.

Remark. If m is odd, one can show that the mth power only has one connected component per row of the original quiver $\Gamma(nm, 1)$. In the even case, there are examples where we obtain one component per row and examples where there are two components per row. For details we refer to [D08].

C. Ducrest has developed a programme to calculate all components of the mth power of $\Gamma(m, n)$ for all $n, m \leq 20$. This programme is available online at
http://www.math.ethz.ch/∼baur/algo/
and the documentation explaining how it works is [D208]. We hope that this programme will help us finding the complete answer for m even.

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